DAY 2

STAN WORKSHOP

RECAP

RECAP

- Ran multiple Stan programs!
- Wrote multiple Stan programs.

  Non-analytic; can only be solved with MCMC.
  Last model isn't found in any package.

- Started learning a new language.

  We conflated statistical modeling and the language.
  Stan language is really flexible.
  For learning: Time helps. Practice helps.

TODAY: MORE ADVANCED STAN

- Iterate over statistical models

- Hierarchical models and non-centered reparameterization

- Exposing functions

HIERARCHICAL MODELS

8 SCHOOLS

- Educational Testing Service studied effect of coaching on SAT scores
- No prior belief any one program was
  - more effective than the others
  - more similar to others

8 SCHOOLS: DATA

School	Estimated treatment effect	Standard error of treatment effect
A	28	15
B	8	10
C	-3	16
D	7	11
E	-1	9
F	1	11
G	18	10
H	12	18

READ DATA

From: https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started

In R:

```
schools_dat <- list(J = 8,
                    y = c(28,  8, -3,  7, -1,  1, 18, 12),
                    sigma = c(15, 10, 16, 11,  9, 11, 10, 18))
```

Stan program:

```
data {
  int J;
  real y[J];
  real<lower = 0> sigma[J];
}
...
```

MODEL 1: COMPLETE POOLING

- Key assumptions
  - Normal likelihood (true for this set of models)
  - All programs have the same effect
    (complete pooling of the effect across schools)
- Hint: add a parameter. This will be interpreted as the coaching effect.

  real theta;

- Write the model.

MODEL 1: COMPLETE POOLING

```
data {
  int J;
  real y[J];
  real<lower = 0> sigma[J];
}
parameters {
  real theta;
}
model {
  y ~ normal(theta, sigma);
}
```

MODEL 1: COMPLETE POOLING

- Run using RStan
- ```
  fit1 <- stan("eight_schools_1.stan",
               data = schools_dat)
  ```

```
> fit1
Inference for Stan model: eight_schools_1.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

       mean se_mean   sd  2.5%   25%   50%   75% 97.5% n_eff Rhat
theta  7.87    0.11 4.07 -0.24  5.22  7.94 10.62 15.74  1465    1
lp__  -2.85    0.02 0.72 -4.85 -3.02 -2.58 -2.40 -2.35  1445    1

Samples were drawn using NUTS(diag_e) at Wed Jun  1 00:57:15 2016.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```
```

MODEL 2: NO POOLING

- Key assumptions
  - Each program has its own, independent effect.

- Hint: instead of one parameter, we have one for each school.

  real theta[J];

- Write the model.

MODEL 2: NO POOLING

```
data {
  int J;
  real y[J];
  real<lower = 0> sigma[J];
}
parameters {
  real theta[J];
}
model {
  y ~ normal(theta, sigma);
}
```

MODEL 2: NO POOLING

```
> fit2
Inference for Stan model: eight_schools_2.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

          mean se_mean    sd   2.5%    25%   50%   75% 97.5% n_eff Rhat
theta[1] 27.70    0.24 14.96  -1.65  17.47 27.82 37.78 56.54  3752    1
theta[2]  7.93    0.17 10.00 -12.17   1.36  8.00 14.72 26.98  3500    1
theta[3] -2.80    0.27 16.56 -35.14 -14.28 -2.46  8.50 29.47  3712    1
theta[4]  6.87    0.18 11.08 -14.55  -0.72  6.69 14.46 28.19  3859    1
theta[5] -0.95    0.15  9.01 -18.51  -7.25 -0.88  5.41 16.28  3690    1
theta[6]  1.16    0.18 11.19 -20.46  -6.41  1.12  8.61 23.59  3724    1
theta[7] 17.64    0.18 10.30  -2.46  10.69 17.68 24.83 37.25  3296    1
theta[8] 11.62    0.29 17.67 -22.80  -0.47 11.53 23.72 45.82  3735    1
lp__     -4.07    0.04  2.02  -9.00  -5.17 -3.71 -2.64 -1.17  2057    1

Samples were drawn using NUTS(diag_e) at Wed Jun  1 01:01:56 2016.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

MODEL 3: PARTIAL POOLING WITH FIXED HYPERPARAMETER

- Key assumptions
  - Hierarchical model on school effects:
    Schools are similar to each other, but not exactly the same
  - Determined by hyperparameter, tau.
- Hint:
  - add new data: real<lower = 0> tau;
  - add new parameter: real mu;
- Write the model.

MODEL 3: PARTIAL POOLING WITH FIXED HYPERPARAMETER

```
data {
  int J;
  real y[J];
  real<lower = 0> sigma[J];
  real<lower = 0> tau;
}
parameters {
  real theta[J];
  real mu;
}
model {
  theta ~ normal(mu, tau);
  y ~ normal(theta, sigma);
}
```

MODEL 3: PARTIAL POOLING WITH FIXED HYPERPARAMETER, TAU = 25

```
> fit3
Inference for Stan model: eight_schools_3.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

          mean se_mean    sd   2.5%   25%   50%   75% 97.5% n_eff Rhat
theta[1] 22.73    0.22 13.18  -3.10 13.91 22.82 31.46 48.13  3664    1
theta[2]  8.04    0.15  9.57 -10.27  1.58  7.94 14.39 27.30  3855    1
theta[3]  0.35    0.23 14.05 -25.89 -9.71  0.25  9.89 28.20  3826    1
theta[4]  7.32    0.16 10.09 -12.09  0.44  7.41 14.39 26.73  4000    1
theta[5]  0.05    0.14  8.60 -16.72 -5.76 -0.03  5.70 17.51  3878    1
theta[6]  2.21    0.16 10.01 -17.48 -4.63  2.23  9.07 21.46  3768    1
theta[7] 16.63    0.15  9.47  -1.92 10.37 16.58 23.05 34.67  3993    1
theta[8] 10.72    0.26 15.13 -18.11  0.39 10.60 21.09 39.70  3514    1
mu        8.63    0.17  9.93 -10.41  1.76  8.69 15.41 28.03  3496    1
lp__     -5.00    0.05  2.10 -10.06 -6.19 -4.67 -3.45 -1.83  1624    1

Samples were drawn using NUTS(diag_e) at Wed Jun  1 01:12:02 2016.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

MODEL 3: PARTIAL POOLING WITH FIXED HYPERPARAMETER, TAU = 0.1

```
> fit3
Inference for Stan model: eight_schools_3.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

          mean se_mean   sd   2.5%   25%   50%   75% 97.5% n_eff Rhat
theta[1]  7.95    0.31 3.98   0.77  5.05  7.64 10.76 15.87   170 1.03
theta[2]  7.95    0.31 3.98   0.81  5.03  7.60 10.81 15.77   170 1.03
theta[3]  7.95    0.30 3.98   0.79  5.04  7.63 10.77 15.83   170 1.03
theta[4]  7.94    0.31 3.98   0.76  5.06  7.64 10.79 15.84   170 1.03
theta[5]  7.94    0.30 3.98   0.76  5.04  7.65 10.77 15.84   170 1.03
theta[6]  7.94    0.31 3.98   0.76  5.03  7.63 10.78 15.79   170 1.03
theta[7]  7.95    0.31 3.98   0.74  5.05  7.62 10.80 15.80   170 1.03
theta[8]  7.95    0.31 3.98   0.72  5.05  7.63 10.80 15.78   170 1.03
mu        7.95    0.31 3.98   0.75  5.02  7.60 10.77 15.82   170 1.03
lp__     -6.80    0.06 2.09 -11.84 -7.91 -6.46 -5.28 -3.69  1100 1.00

Samples were drawn using NUTS(diag_e) at Wed Jun  1 01:13:53 2016.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

MODEL 3: PARTIAL POOLING WITH FIXED HYPERPARAMETER, TAU = 1000

```
> fit3
Inference for Stan model: eight_schools_3.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

          mean se_mean     sd    2.5%     25%   50%    75%  97.5% n_eff Rhat
theta[1] 28.31    0.24  15.15   -1.78   18.33 28.18  38.25  58.75  4000    1
theta[2]  8.20    0.17  10.04  -11.69    1.64  8.26  14.95  27.42  3575    1
theta[3] -2.74    0.25  16.02  -33.62  -13.52 -2.88   8.21  28.98  4000    1
theta[4]  6.96    0.17  10.82  -14.10   -0.23  6.75  13.91  28.85  4000    1
theta[5] -1.00    0.15   9.20  -19.41   -7.08 -0.94   5.11  17.39  3530    1
theta[6]  0.87    0.18  11.46  -21.78   -6.76  0.81   8.67  23.05  4000    1
theta[7] 18.07    0.16   9.94   -1.66   11.46 18.06  24.75  37.23  4000    1
theta[8] 12.15    0.28  17.75  -22.30    0.21 11.65  24.43  46.33  3980    1
mu        5.04    5.79 351.04 -673.13 -229.53 11.84 239.87 676.59  3679    1
lp__     -4.54    0.05   2.17   -9.47   -5.76 -4.18  -2.96  -1.36  1631    1

Samples were drawn using NUTS(diag_e) at Wed Jun  1 01:15:48 2016.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

EFFECT OF TAU ON THETA

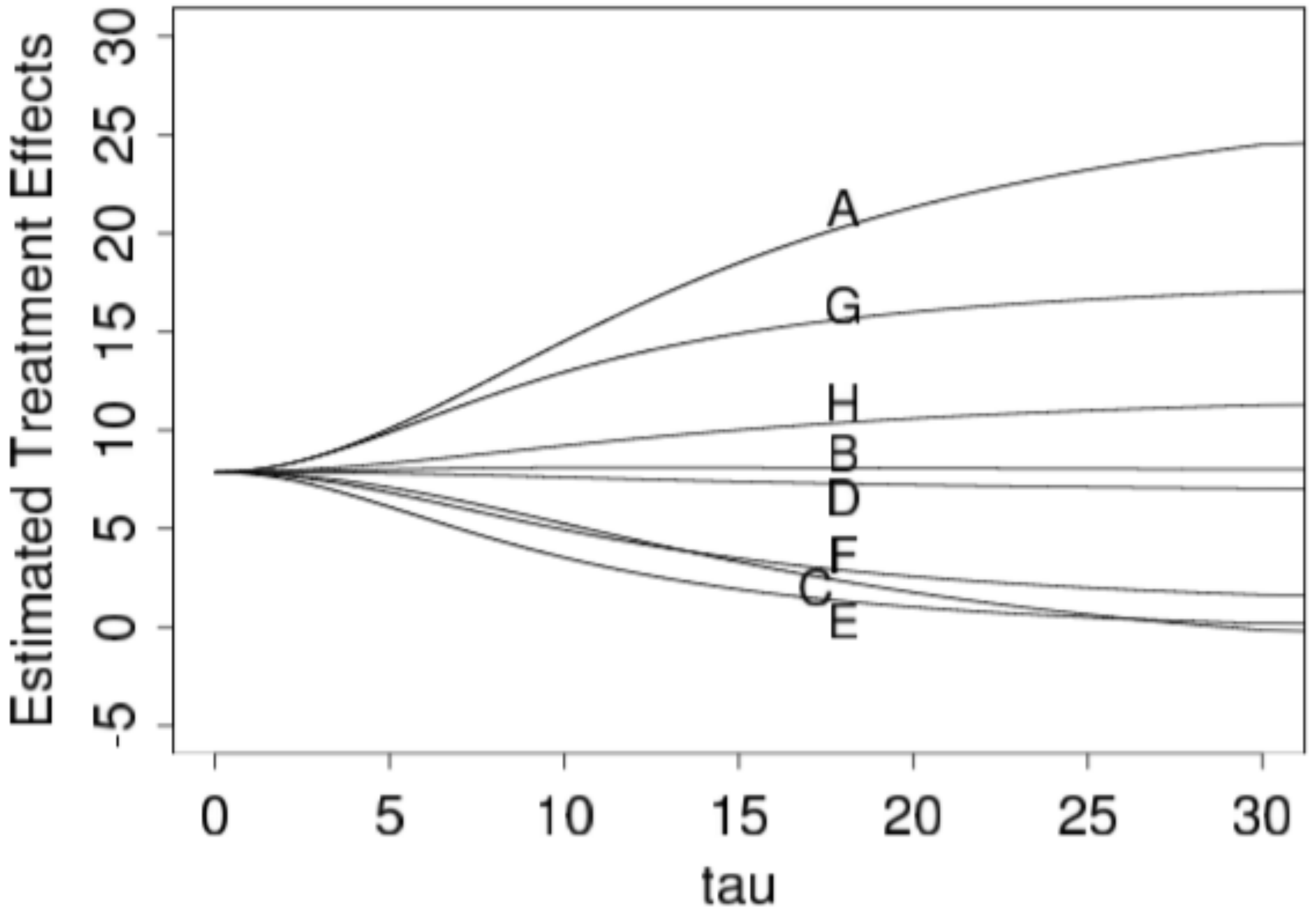


Figure 5.6 *Conditional posterior means of treatment effects,* $E(\theta_j|\tau, y)$, *as functions of the between-school standard deviation* τ, *for the educational testing example. The line for school* C *crosses the lines for* E *and* F *because* C *has a higher measurement error (see Table 5.2) and its estimate is therefore shrunk more strongly toward the overall mean in the Bayesian analysis.*

WOULDN'T IT BE NICE TO LEARN TAU?

- Amount of pooling determined by data!
- Easy to specify: change tau to a parameter.
- Problems when running:

```
Warning messages:

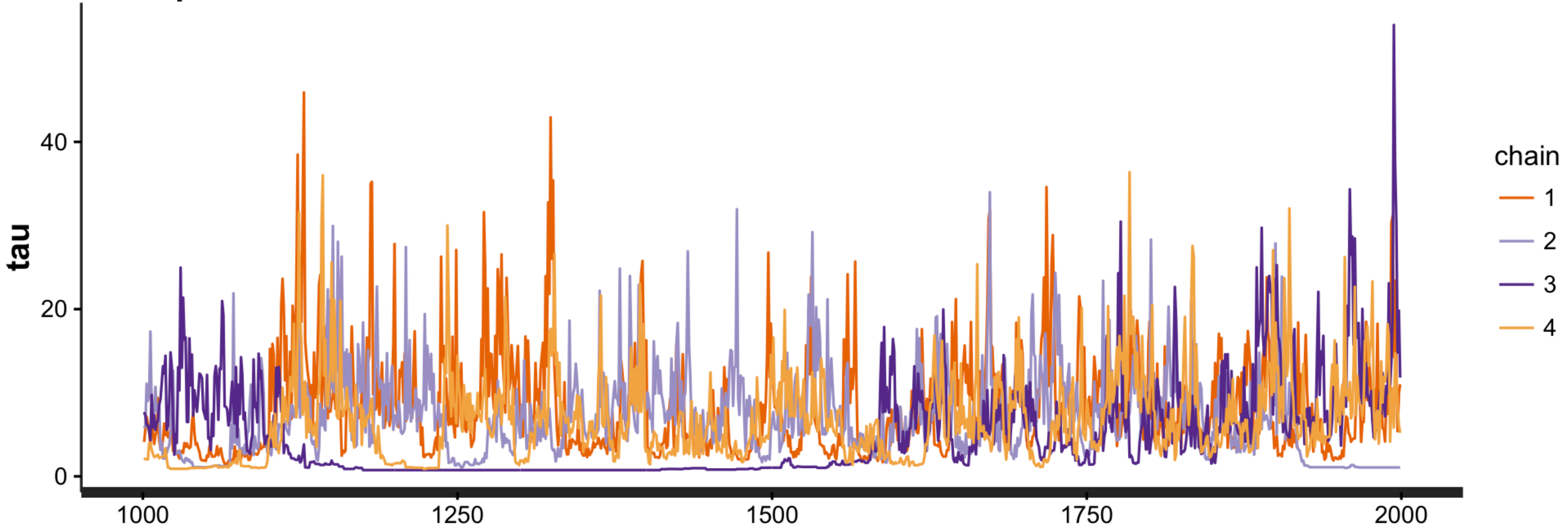
1: There were 124 divergent transitions after
warmup. Increasing adapt_delta above 0.8 may
help.

2: Examine the pairs() plot to diagnose sampling
problems
```

PROBLEM: DIVERGENT TRANSITIONS

- **Do not** ignore divergent transitions
  - Part of the sample space is ill-behaved
  - MCMC estimates will be biased

- `traceplot(fit, “tau”)`



PROBLEM: DIVERGENT TRANSITIONS

- **Do not** ignore divergent transitions
  - Part of the sample space is ill-behaved
  - MCMC estimates will be biased

- Fixes

1. super-easy: set adapt_delta higher

```
fit4 <- stan(fit = fit4, data = schools_dat,
             control=list(adapt_delta = 0.9))
```

2. easy: reparameterize (next section)

QUICK ASIDE

- Why doesn't MLE work for the hierarchical model?

QUICK ASIDE

- Why doesn't MLE work for this hierarchical model?

```
> optimizing(fit4@stanmodel, data = schools_dat)
STAN OPTIMIZATION COMMAND (LBFGS)
init = random
save_iterations = 1
init_alpha = 0.001
tol_obj = 1e-12
tol_grad = 1e-08
tol_param = 1e-08
tol_rel_obj = 10000
tol_rel_grad = 1e+07
history_size = 5
seed = 286152403
initial log joint probability = -119.833
Optimization terminated with error: 
  Line search failed to achieve a sufficient decrease, no more progress can be made
$par
    theta[1]     theta[2]     theta[3]     theta[4]     theta[5]     theta[6]     theta[7]     theta[8]           mu          tau 
1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 2.205697e-16 

$value
[1] 283.0284
```

QUICK ASIDE

- Why doesn't MLE work for this hierarchical model?

```
> optimizing(fit4@stanmodel, data = schools_dat)
STAN OPTIMIZATION COMMAND (LBFGS)
init = random
save_iterations = 1
init_alpha = 0.001
tol_obj = 1e-12
tol_grad = 1e-08
tol_param = 1e-08
tol_rel_obj = 10000
tol_rel_grad = 1e+07
history_size = 5
seed = 286152403
initial log joint probability = -119.833
Optimization terminated with error:
  Line search failed to achieve a sufficient decrease, no more progress can be made
$par
    theta[1]     theta[2]     theta[3]     theta[4]     theta[5]     theta[6]     theta[7]     theta[8]           mu          tau
1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 1.945095e+00 2.205697e-16

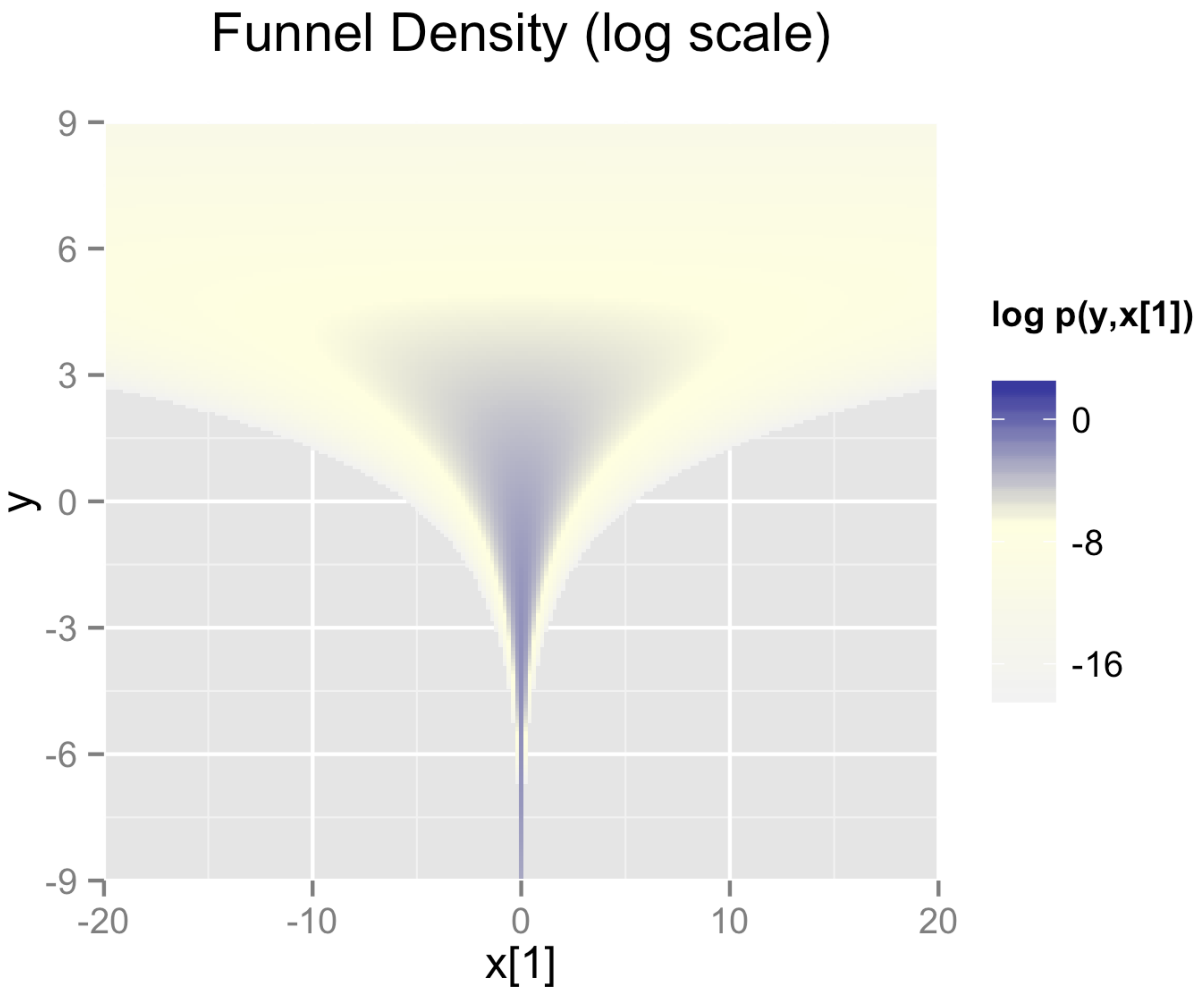
$value
[1] 283.0284
```

NON-CENTERED REPARAMETERIZATION

FUNNEL

- $y \in \mathbb{R}$
- $x \in \mathbb{R}^9$

$$p(y, x) = \text{Normal}(y|0, 3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0, \exp(y/2))$$



WHEN DO YOU SEE THIS?

- Hierarchical models
- Variance parameters go to 0, all parameters shrink
  Variance parameters get large, all parameters spread
- Trick to handle low data situations
- Called non-centered parameterization aka the Matt trick …

STEPS

1. Add new parameter, *_raw.
2. Move original parameter to transformed parameters block.
3. Assign transformation of *_raw to original parameter.
4. Put Normal(0, 1) prior on *_raw.

CENTERED FUNNEL

- Easy to write in Stan
- Run. See any problems?

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 1

- Add new parameter, *_raw.

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 1

- Add new parameter, *_raw.

```
parameters {
  real y;
  vector[9] x;
  real y_raw;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 2

- Move original parameter to transformed parameters block.

```
parameters {
  real y;
  vector[9] x;
  real y_raw;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 2

- Move original parameter to transformed parameters block.

```
parameters {
  vector[9] x;
  real y_raw;
}
transformed parameters {
  real y;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 3

- Assign transformation of *_raw to original parameter.

```
parameters {
  vector[9] x;
  real y_raw;
}
transformed parameters {
  real y;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 3

- Assign transformation of *_raw to original parameter.

```
parameters {
  vector[9] x;
  real y_raw;
}
transformed parameters {
  real y;
  y <- 3 * y_raw;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 4

- Put Normal(0, 1) prior on *_raw.

```
parameters {
  vector[9] x;
  real y_raw;
}
transformed parameters {
  real y;
  y <- 3 * y_raw;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL: STEP 4

- Put Normal(0, 1) prior on *_raw.

```
parameters {
  vector[9] x;
  real y_raw;
}
transformed parameters {
  real y;
  y <- 3 * y_raw;
}
model {
  y_raw ~ normal(0, 1);
  x ~ normal(0, exp(y/2));
}
```

NON-CENTERED FUNNEL

- Repeat for xs.
- Steps:

1. Add new parameter, *_raw.
2. Move original parameter to transformed parameters block.
3. Assign transformation of *_raw to original parameter.
4. Put Normal(0, 1) prior on *_raw.

NON-CENTERED FUNNEL

```
parameters {
  real y_raw;
  vector[9] x_raw;
}
transformed parameters {
  real y;
  vector[9] x;

  y <- 3.0 * y_raw;
  x <- exp(y/2) * x_raw;
}
model {
  y_raw ~ normal(0, 1);
  x_raw ~ normal(0, 1);
}
```

CENTERED VS NON-CENTERED

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0, 3);
  x ~ normal(0, exp(y/2));
}
```

```
parameters {
  real y_raw;
  vector[9] x_raw;
}
transformed parameters {
  real y;
  vector[9] x;

  y <- 3.0 * y_raw;
  x <- exp(y/2) * x_raw;
}
model {
  y_raw ~ normal(0, 1);
  x_raw ~ normal(0, 1);
}
```

EIGHT SCHOOLS

REPARAMETERIZE EIGHT SCHOOLS

- Here's the centered parameterization.

```
data {
  int J;
  real y[J];
  real<lower = 0> sigma[J];
}
parameters {
  real theta[J];
  real mu;
  real<lower = 0> tau;
}
model {
  theta ~ normal(mu, tau);
  y ~ normal(theta, sigma);
}
```

NON-CENTERED REPARAMETERIZATION

- Follow steps:
  1. Add new parameter, *_raw.
  2. Move original parameter to transformed parameters block.
  3. Assign transformation of *_raw to original parameter.
  4. Put Normal(0, 1) prior on *_raw.

MODEL 5: HIERARCHICAL, NON-CENTERED REPARAMETERIZATION

```
data {
  int J;
  real y[J];
  real<lower = 0> sigma[J];
}
parameters {
  real theta_raw[J];
  real mu;
  real<lower = 0> tau;
}
transformed parameters {
  real theta[J];
  for (j in 1:J)
    theta <- mu + tau * theta_raw[j];
}
model {
  theta_raw ~ normal(0, 1);
  y ~ normal(theta, sigma);
}
```

MODEL 5: HIERARCHICAL, NON-CENTERED REPARAMETERIZATION

```
> fit5
Inference for Stan model: eight_schools_5.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

              mean se_mean   sd   2.5%   25%   50%   75% 97.5% n_eff Rhat
theta_raw[1]  0.43    0.02 0.95  -1.56 -0.19  0.47  1.09  2.23  1988 1.00
theta_raw[2]  0.00    0.02 0.89  -1.84 -0.57  0.00  0.59  1.75  3048 1.00
theta_raw[3] -0.22    0.02 0.95  -2.01 -0.87 -0.23  0.42  1.64  3282 1.00
theta_raw[4] -0.03    0.02 0.86  -1.71 -0.60 -0.03  0.53  1.70  2909 1.00
theta_raw[5] -0.37    0.02 0.84  -1.98 -0.93 -0.38  0.18  1.33  2573 1.00
theta_raw[6] -0.23    0.02 0.87  -1.97 -0.79 -0.23  0.35  1.52  2823 1.00
theta_raw[7]  0.36    0.02 0.88  -1.53 -0.19  0.39  0.95  1.99  2945 1.00
theta_raw[8]  0.06    0.02 0.95  -1.82 -0.55  0.06  0.66  1.93  2659 1.00
mu            7.82    0.15 5.04  -2.25  4.72  7.72 11.02 17.54  1183 1.00
tau           6.68    0.18 5.38   0.28  2.69  5.49  9.25 19.99   928 1.00
theta[1]     11.70    0.27 8.73  -2.02  6.12 10.38 15.91 33.14  1020 1.00
theta[2]      7.96    0.11 6.28  -4.12  3.96  7.86 11.83 20.89  3092 1.00
theta[3]      5.89    0.16 7.86 -11.69  1.64  6.32 10.75 20.51  2519 1.00
theta[4]      7.53    0.13 6.46  -5.81  3.77  7.56 11.49 20.47  2622 1.00
theta[5]      4.89    0.12 6.14  -8.70  1.31  5.40  9.00 16.09  2766 1.00
theta[6]      5.99    0.12 6.62  -8.25  2.08  6.34 10.18 18.03  2943 1.00
theta[7]     10.70    0.14 6.78  -1.41  6.26 10.06 14.72 25.91  2374 1.00
theta[8]      8.34    0.18 8.00  -7.21  3.70  7.98 12.75 25.99  1961 1.00
lp__         -4.77    0.09 2.65 -10.77 -6.31 -4.44 -2.90 -0.40   972 1.01

Samples were drawn using NUTS(diag_e) at Wed Jun  1 01:47:53 2016.
```

TAKEAWAYS

- Easy to express hierarchical models in Stan!

- Discussed how to get around some divergent transitions
  - Increase adapt_delta
  - Non-centered reparameterization

FUNCTIONS

STAN FUNCTIONS IN R!

1. Write Stan program with a function.
2. In R:

```
cppcode <- stanc("functions.stan")
expose_stan_functions(cppcode)
```

OR

```
fit <- stanc("functions.stan")
expose_stan_functions(fit)
```

ACCESSING PARTS OF THE STAN PROGRAM IN R

- get_num_upars(fit)

- log_prob(fit, upars)
- grad_log_prob(fit, upars)
- constrain_pars(fit, upars)

- unconstrain_pars(fit, pars)