Stan

Statistical Inference Made Easy

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Stan

Section 1.
Bayesian Inference

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Warmup Exercise I

Sample Variation

Repeated i.i.d. Trials

- Suppose we repeatedly generate a random outcome from among several potential outcomes
- Suppose the outcome chances are the same each time
  - i.e., outcomes are independent and identically distributed (i.i.d.)
- For example, spin a fair spinner (without cheating), such as one from *Family Cricket*.

Image source: http://replaycricket.com/2010/10/29/family-cricket/

Repeated i.i.d. Binary Trials

- Suppose the outcome is binary and assigned to 0 or 1; e.g.,
  - 20% chance of outcome 1: *ball in play*
  - 80% chance of outcome 0: *ball* not *in play*
- Consider different numbers of bowls delivered.
- How will proportion of successes in sample differ?

Simulating i.i.d. Binary Trials

- R Code: `rbinom(10, N, 0.2) / N`
  - **10 bowls** (10% to 50% success rate)

    2 3 5 2 4 1 2 2 1 1
  - **100 bowls** (16% to 26% success rate)

    26 18 23 17 21 16 21 15 21 26
  - **1000 bowls** (18% to 22% success rate)

    181 212 175 213 216 179 223 198 188 194
  - **10,000 bowls** (19.3% to 20.3% success rate)

    2029 1955 1981 1980 2001 2014 1931 1982 1989 2020

Simple Point Estimation

- Estimate chance of success θ by proportion of successes:

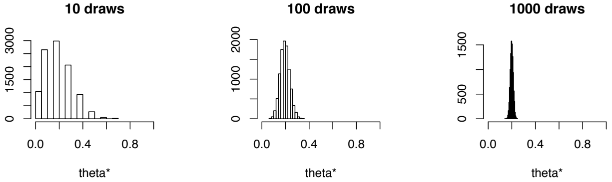
$$\theta^* = \frac{\text{successes}}{\text{attempts}}$$

- Simulation shows accuracy depends on the amount of data.
- Statistical inference includes quantifying uncertainty.
- Bayesian statistics is about using uncertainty in inference.

Estimation Uncertainty

- Simulation of estimate variation due to sampling
- *not* a Bayesian posterior

```
> num_sims <- 10000;    N <- 100;    theta <- 0.2;
> hist(rbinom(num_sims, N, theta) / N,
       main=sprintf("%d draws",N), xlab="theta*");
```



Estimator Bias

- **Bias:** expected difference of estimate from true value
- Continuing previous example

```
> sims <- rbinom(10000, 1000, 0.2) / 1000
> mean(sims)
[1] 0.2002536
```

- Value of 0.2 is estimate of expectation
- Shows this estimator is *unbiased*

Simple Point Estimation (cont.)

- **Central Limit Theorem:** *expected* error in θ^* goes down as

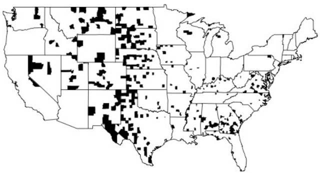
$$\frac{1}{\sqrt{N}}$$

- Each decimal place of accuracy requires $100\times$ more samples.
- Width of confidence intervals shrinks at the same rate.
- Can also use theory to show this estimator is unbiased.

Pop Quiz! Cancer Clusters

- Why do lowest and highest cancer clusters look so similar?

Lowest kidney cancer death rates



Highest kidney cancer death rates

Image from Gelman et al., *Bayesian Data Analysis, 3rd Edition* (2013)

Pop Quiz Answer

- Hint: mix earlier simulations of repeated i.i.d. trials with 20% success and sort:

1/10	1/10	1/10	15/100	16/100
17/100	175/1000	179/1000	18/100	181/1000
188/1000	194/1000	198/1000	2/10	2/10
2/10	2/10	21/100	21/100	21/100
212/1000	213/1000	216/1000	223/1000	23/100
26/100	26/100	3/10	4/10	5/10

- More variation in observed rates with smaller sample sizes

- *Answer*: High cancer and low cancer counties are small populations

Warmup Exercise II

Maximum Likelihood Estimation

Observations, Counterfactuals, and Random Variables

- Assume we observe data $y = y_1, \ldots, y_N$
- Statistical modeling assumes even though y is observed, the values could have been different
- John Stuart Mill first characterized this **counterfactual** nature of statistical modeling in:

  A System of Logic, Ratiocinative and Inductive (1843)
- In measure-theoretic language, y is a **random variable**

Likelihood Functions

- A **likelihood function** is a probability function (density, mass, or mixed)

$$p(y|\theta, x),$$

  where

  - θ is a vector of **parameters**,
  - x is some fixed **unmodeled data** (e.g., regression predictors or "features"),
  - y is some fixed **modeled data** (e.g., observations)

- considered as a function $\mathcal{L}(\theta)$ of θ for fixed x and y.
- can think of as a generative process for $y$how data y is generated

Maximum Likelihood Estimation

- **Estimate** parameters θ given observations y.
- Maximum likelihood estimation (MLE) chooses estimate that maximizes the likelihood function, i.e.,

$$\theta^* = \arg\max_\theta \mathcal{L}(\theta) = \arg\max_\theta p(y|\theta, x)$$

- This function of $\mathcal{L}$ and y (and x) is called an **estimator**

Example of MLE

- The frequency-based estimate

$$\theta^* = \frac{1}{N} \sum_{n=1}^{N} y_n,$$

is the observed rate of “success” (outcome 1) observations.

- This is the MLE for the model

$$p(y|\theta) \;=\; \prod_{n=1}^{N} p(y_n|\theta) \;=\; \prod_{n=1}^{N} \text{Bernoulli}(y_n|\theta)$$

where for $u \in \{0, 1\}$,

$$\text{Bernoulli}(u|\theta) = \begin{cases} \theta & \text{if } u = 1 \\ 1 - \theta & \text{if } u = 0 \end{cases}$$

Example of MLE (cont.)

- First modeling *assumption* is that data are i.i.d.,

$$p(y|\theta) = \prod_{n=1}^{N} p(y_n|\theta)$$

- Second modeling *assumption* is form of likelihood,

$$p(y_n|\theta) = \text{Bernoulli}(y_n|\theta)$$

Example of MLE (cont.)

- The frequency-based estimate is the MLE
- First derivative is zero (indicating min or max),

$$\mathcal{L}_y'(\theta^*) = 0,$$

- Second derivative is negative (indicating max),

$$\mathcal{L}_y''(\theta^*) < 0.$$

MLEs can be Dangerous!

- Recall the cancer cluster example
- Accuracy is low with small counts
- What we need are hierarchical models (stay tuned)

Part I
Bayesian Inference

Bayesian Data Analysis

- "By Bayesian data analysis, we mean practical methods for making inferences from data using probability models for quantities we observe and about which we wish to learn."
- "The essential characteristic of Bayesian methods is their **explict use of probability for quantifying uncertainty** in inferences based on statistical analysis."

Gelman et al., *Bayesian Data Analysis*, 3rd edition, 2013

Bayesian Methodology

- Set up **full probability model**
  - for all observable & unobservable quantities
  - consistent w. problem knowledge & data collection
- **Condition** on observed data
  - to caclulate posterior probability of unobserved quantities (e.g., parameters, predictions, missing data)
- **Evaluate**
  - model fit and implications of posterior
- **Repeat** as necessary

Ibid.

Where do Models Come from?

- Sometimes model comes first, based on substantive considerations
  - toxicology, economics, ecology, . . .
- Sometimes model chosen based on data collection
  - traditional statistics of surveys and experiments
- Other times the data comes first
  - observational studies, meta-analysis, . . .
- Usually its a mix

(Donald) Rubin's Philosophy

- All statistics is inference about missing data
- Question 1: What would you do if you had all the data?
- Question 2: What were you doing before you had any data?

(as relayed in course notes by Andrew Gelman)

Model Checking

- Do the inferences make sense?
  - are parameter values consistent with model's prior?
  - does simulating from parameter values produce reasoable fake data?
  - are marginal predictions consistent with the data?
- Do predictions and event probabilities for new data make sense?
- ***Not***: Is the model true?
- ***Not***: What is Pr[model is true]?
- ***Not***: Can we "reject" the model?

Model Improvement

- Expanding the model
  - hierarchical and multilevel structure . . .
  - more flexible distributions (overdispersion, covariance)
  - more structure (geospatial, time series)
  - more modeling of measurement methods and errors
  - . . .
- Including more data
  - breadth (more predictors or kinds of observations)
  - depth (more observations)

Using Bayesian Inference

- Finds parameters consistent with prior info and data*
  - * if such agreement is possible
- Automatically includes uncertainty and variability
- Inferences can be plugged in directly
  - risk assesment
  - decision analysis

Notation for Basic Quantities

- **Basic Quantities**
  - y: observed data
  - θ: parameters (and other unobserved quantities)
  - x: constants, predictors for conditional (aka "discriminative") models
- **Basic Predictive Quantities**
  - $\tilde{y}$: unknown, potentially observable quantities
  - $\tilde{x}$: constants, predictors for unknown quantities

Naming Conventions

- **Joint**: $p(y, \theta)$
- **Sampling / Likelihood**: $p(y|\theta)$
  - Sampling is function of y with θ fixed (prob function)
  - Likelihood is function of θ with y fixed (*not* prob function)
- **Prior**: $p(\theta)$
- **Posterior**: $p(\theta|y)$
- **Data Marginal (Evidence)**: $p(y)$
- **Posterior Predictive**: $p(\tilde{y}|y)$

Bayes's Rule for Posterior

$$
\begin{aligned}
p(\theta|y) &= \frac{p(y,\theta)}{p(y)} && \text{[def of conditional]} \\
&= \frac{p(y|\theta)\, p(\theta)}{p(y)} && \text{[chain rule]} \\
&= \frac{p(y|\theta)\, p(\theta)}{\int_\Theta p(y,\theta')\; d\theta'} && \text{[law of total prob]} \\
&= \frac{p(y|\theta)\, p(\theta)}{\int_\Theta p(y|\theta')\, p(\theta')\; d\theta'} && \text{[chain rule]}
\end{aligned}
$$

- *Inversion:* Final result depends only on sampling distribution (likelihood) $p(y|\theta)$ and prior $p(\theta)$

Bayes's Rule up to Proportion

- If data y is fixed, then

$$
\begin{aligned}
p(\theta|y) &= \frac{p(y|\theta)\,p(\theta)}{p(y)} \\
&\propto p(y|\theta)\,p(\theta) \\
&= p(y,\theta)
\end{aligned}
$$

- Posterior proportional to likelihood times prior
- Equivalently, posterior proportional to joint
- The nasty integral for data marginal $p(y)$ goes away

Posterior Predictive Distribution

- Predict new data $\tilde{y}$ based on observed data y
- Marginalize out parameters from posterior

$$p(\tilde{y}|y) = \int_{\Theta} p(\tilde{y}|\theta)\, p(\theta|y)\, d\theta.$$

- Averages predictions $p(\tilde{y}|\theta)$, weight by posterior $p(\theta|y)$
  - $\Theta = \{\theta \mid p(\theta|y) > 0\}$ is support of $p(\theta|y)$
- Allows continuous, discrete, or mixed parameters
  - integral notation shorthand for sums and/or integrals

Event Probabilities

- Recall that an event A is a collection of outcomes
- Suppose event A is determined by indicator on parameters

$$f(\theta) = \begin{cases} 1 & \text{if } \theta \in A \\ 0 & \text{if } \theta \notin A \end{cases}$$

- e.g., $f(\theta) = \mathrm{I}(\theta_1 > \theta_2)$ for $\mathrm{Pr}[\theta_1 > \theta_2 \mid y]$
- Bayesian event probabilities calculate posterior mass

$$\mathrm{Pr}[A] = \int_{\Theta} f(\theta)\, p(\theta|y)\, d\theta.$$

- Not frequentist, because involves parameter probabilities

Example I

Male Birth Ratio

Laplace’s Data and Problems

- Laplace’s data on live births in Paris from 1745–1770:

sex	*live births*
female	241 945
male	251 527

- Question 1 (Event Probability)
  Is a boy more likely to be born than a girl?
- Question 2 (Estimate)
  What is the birth rate of boys vs. girls?
- Bayes formulated the basic binomial model
- Laplace solved the integral

Binomial Distribution

- Binomial distribution is number of successes y in N i.i.d. Bernoulli trials with chance of success θ
- If $y_1, \ldots, y_N \sim \text{Bernoulli}(\theta)$,

  then $(y_1 + \cdots + y_N) \sim \text{Binomial}(N, \theta)$
- The analytic form is

$$\text{Binomial}(y|N, \theta) = \binom{N}{y} \theta^y (1 - \theta)^{N-y}$$

  where the binomial coefficient normalizes for permutations (i.e., which subset of n has $y_n = 1$),

$$\binom{N}{y} = \frac{N!}{y!\,(N-y)!}$$

Bayes's Binomial Model

- Data
  - y: total number of male live births (data: 241 945)
  - N : total number of live births (data: 493 472)
- Parameter
  - $\theta \in (0, 1)$: proportion of male live births
- Likelihood

$$p(y|N, \theta) \; = \; \text{Binomial}(y|N, \theta) \; = \; \binom{N}{y} \theta^y (1 - \theta)^{N-y}$$

- Prior

$$p(\theta) \; = \; \text{Uniform}(\theta \,|\, 0, 1) \; = \; 1$$

Detour: Beta Distribution

- For parameters $\alpha, \beta > 0$ and $\theta \in (0, 1)$,

$$\text{Beta}(\theta|\alpha, \beta) \;\propto\; \theta^{\alpha-1}\,(1-\theta)^{\beta-1}$$

- Euler's Beta function is used to normalize,

$$\text{B}(\alpha, \beta) \;=\; \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du \;=\; \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

  so that

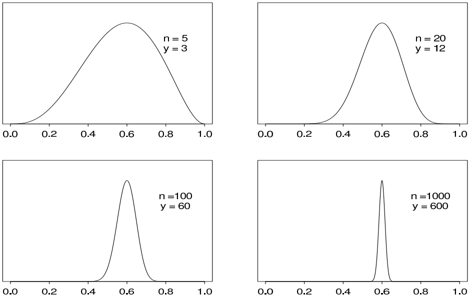
$$\text{Beta}(\theta|\alpha, \beta) \;=\; \frac{1}{\text{B}(\alpha, \beta)}\,\theta^{\alpha-1}\,(1-\theta)^{\beta-1}$$

- Note: $\text{Beta}(\theta|1, 1) = \text{Uniform}(\theta|0, 1)$

- Note: $\Gamma()$ is continuous generalization of factorial

Beta Distribution — Examples

- Unnormalized posterior density assuming uniform prior and y successes out of n trials (all with mean 0.6).



Gelman et al. (2013) *Bayesian Data Analysis*, 3rd Edition.

Laplace Turns the Crank

- From Bayes's rule, the posterior is

$$p(\theta|y,N) \;=\; \frac{\text{Binomial}(y|N,\theta)\,\text{Uniform}(\theta|0,1)}{\int_{\Theta}\text{Binomial}(y|N,\theta')\,p(\theta')\;d\theta'}$$

- Laplace calculated the posterior analytically

$$p(\theta|y,N) \;=\; \text{Beta}(\theta\,|\,y+1,\;N-y+1).$$

Estimation

- Posterior is Beta$(\theta \mid 1 + 241\,945,\ 1 + 251\,527)$
- Posterior mean:

$$\frac{1 + 241\,945}{1 + 241\,945 + 1 + 251\,527} \approx 0.490291\mathbf{3}$$

- Maximum likelihood estimate same as posterior mode (because of uniform prior)

$$\frac{241\,945}{241\,945 + 251\,527} \approx 0.490291\mathbf{2}$$

- As number of observations approaches ∞,
  MLE approaches posterior mean

Event Probability Inference

- What is probability that a male live birth is more likely than a female live birth?

$$
\begin{aligned}
\Pr[\theta > 0.5] &= \int_{\Theta} \mathrm{I}[\theta > 0.5]\, p(\theta|y,N)\, d\theta \\
&= \int_{0.5}^{1} p(\theta|y,N)\, d\theta \\
&= 1 - F_{\theta|y,N}(0.5) \\
&\approx 10^{-42}
\end{aligned}
$$

- $\mathrm{I}[\phi] = 1$ if condition ϕ is true and 0 otherwise.
- $F_{\theta|y,N}$ is posterior cumulative distribution function (cdf).

Mathematics vs. Simulation

- Luckily, we don't have to be as good at math as Laplace
- Nowadays, we calculate all these integrals by computer using tools like Stan

> If you wanted to do foundational research in statistics in the mid-twentieth century, you had to be bit of a mathematician, whether you wanted to or not. . . . if you want to do statistical research at the turn of the twenty-first century, you have to be a computer programmer.
>
> —from Andrew's blog

Bayesian "Fisher Exact Test"

- Suppose we observe the following data on handedness

	sinister	*dexter*	TOTAL
male	9 (y_1)	43	52 (N_1)
female	4 (y_2)	44	48 (N_2)

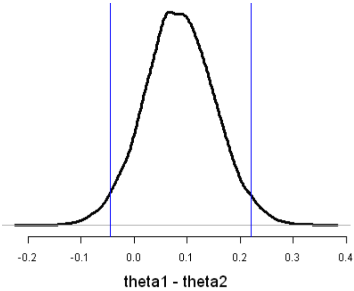
- Assume likelihoods $\text{Binomial}(y_k | N_k, \theta_k)$, uniform priors
- Are men more likely to be lefthanded?

$$\begin{aligned} \Pr[\theta_1 > \theta_2 \mid y, N] &= \int_\Theta \mathrm{I}[\theta_1 > \theta_2]\, p(\theta|y,N)\, d\theta \\ &\approx 0.91 \end{aligned}$$

- Directly interpretable result; *not* a frequentist procedure

Visualizing Posterior Difference

- Plot of posterior difference, $p(\theta_1 - \theta_2 \mid y, N)$ (men - women)



- Vertical bars: central 95% posterior interval $(-0.05, 0.22)$

Technical Interlude

Conjugate Priors

Conjugate Priors

- Family $\mathcal{F}$ is a conjugate prior for family $\mathcal{G}$ if
  - prior in $\mathcal{F}$ and
  - likelihood in $\mathcal{G}$,
  - entails posterior in $\mathcal{F}$
- Before MCMC techniques became practical, Bayesian analysis mostly involved conjugate priors
- Still widely used because analytic solutions are more efficient than MCMC

Beta is Conjugate to Binomial

- Prior: $p(\theta|\alpha,\beta) = \mathrm{Beta}(\theta|\alpha,\beta)$
- Likelihood: $p(y|N,\theta) = \mathrm{Binomial}(y|N,\theta)$
- Posterior:

$$\begin{aligned}
p(\theta|y,N,\alpha,\beta) &\propto p(\theta|\alpha,\beta)\, p(y|N,\theta) \\
&= \mathrm{Beta}(\theta|\alpha,\beta)\, \mathrm{Binomial}(y|N,\theta) \\
&= \frac{1}{\mathrm{B}(\alpha,\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \binom{N}{y} \theta^{y} (1-\theta)^{N-y} \\
&\propto \theta^{y+\alpha-1} (1-\theta)^{N-y+\beta-1} \\
&\propto \mathrm{Beta}(\theta|\alpha+y,\ \beta+(N-y))
\end{aligned}$$

Chaining Updates

- Start with prior Beta$(\theta|\alpha,\beta)$
- Receive binomial data in K statges $(y_1,N_1),\ldots,(y_K,N_K)$
- After (y_1,N_1), posterior is Beta$(\theta|\alpha+y_1,\ \beta+N_1-y_1)$
- Use as prior for (y_2,N_2), with posterior
  Beta$(\theta|\alpha+y_1+y_2,\ \ \beta+(N_1-y_1)+(N_2-y_2))$
- Lather, rinse, repeat, until final posterior
  Beta$(\theta|\alpha+y_1+\cdots+y_K,\ \ \beta+(N_1+\cdots+N_K)-(y_1+\cdots+y_K))$
- Same result as if we'd updated with combined data
  Beta$(y_1+\cdots+y_K,\ \ N_1+\cdots+N_K)$

Part II
(Un-)Bayesian Point Estimation

MAP Estimator

- For a Bayesian model $p(y, \theta) = p(y|\theta)\,p(\theta)$, the max a posteriori (MAP) estimate maximizes the posterior,

$$
\begin{aligned}
\theta^* &= \text{arg max}_\theta \; p(\theta|y) \\
&= \text{arg max}_\theta \; \frac{p(y|\theta)p(\theta)}{p(y)} \\
&= \text{arg max}_\theta \; p(y|\theta)p(\theta). \\
&= \text{arg max}_\theta \; \log p(y|\theta) + \log p(\theta).
\end{aligned}
$$

- *not* Bayesian because it doesn't integrate over uncertainty
- *not* frequentist because of distributions over parameters

MAP and the MLE

- MAP estimate reduces to the MLE if the prior is uniform, i.e.,

$$p(\theta) = c$$

because

$$\begin{aligned} \theta^* &= \text{arg max}_\theta \; p(y|\theta)\, p(\theta) \\ &= \text{arg max}_\theta \; p(y|\theta)\, c \\ &= \text{arg max}_\theta \; p(y|\theta). \end{aligned}$$

Penalized Maximum Likelihood

- The MAP estimate can be made palatable to frequentists via philosophical sleight of hand
- Treat the negative log prior $-\log p(\theta)$ as a "penalty"
- e.g., a Normal$(\theta|\mu,\sigma)$ prior becomes a penalty function

$$\lambda_{\theta,\mu,\sigma} = -\left(\log\sigma + \frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2\right)$$

- Maximize sum of log likelihood and negative penalty

$$\begin{aligned}\theta^* &= \text{arg max}_\theta \ \log p(y|\theta,x) - \lambda_{\theta,\mu,\sigma} \\ &= \text{arg max}_\theta \ \log p(y|\theta,x) + \log p(\theta|\mu,\sigma)\end{aligned}$$

Proper Bayesian Point Estimates

- Choose estimate to minimize some loss function
- To minimize expected squared error (L2 loss), $\mathbb{E}[(\theta - \theta')^2 \mid y]$, use the posterior mean

$$\hat{\theta} = \text{arg min}_{\theta'} \mathbb{E}[(\theta - \theta')^2 \mid y] = \int_{\Theta} \theta \times p(\theta|y)\, d\theta.$$

- To minimize expected absolute error (L1 loss), $\mathbb{E}[|\theta - \theta'|]$, use the posterior median.
- Other loss (utility) functions possible, the study of which falls under decision theory
- All share property of involving full Bayesian inference.

Point Estimates for Inference?

- Common in machine learning to generate a point estimate θ^*, then use it for inference, $p(\tilde{y}|\theta^*)$
- This is **defective** because it

  underestimates uncertainty.

- To properly estimate uncertainty, apply full Bayes
- A major focus of statistics and decision theory is estimating uncertainty in our inferences

Philosophical Interlude

What is Statistics?

Exchangeability

- Roughly, an exchangeable probability function is such that for a sequence of random variables $y = y_1, \ldots, y_N$,

$$p(y) = p(\pi(y))$$

  for every N-permutation π (i.e, a one-to-one mapping of $\{1, \ldots, N\}$)

- i.i.d. implies exchangeability, but not vice-versa

Exchangeability Assumptions

- Models almost always make some kind of exchangeability assumption
- Typically when other knowledge is not available
  - e.g., treat voters as conditionally i.i.d. given their age, sex, income, education level, religous affiliation, and state of residence
  - But voters have many more properties (hair color, height, profession, employment status, marital status, car ownership, gun ownership, etc.)
  - Missing predictors introduce additional error (on top of measurement error)

Random Parameters: Doxastic or Epistemic?

- Bayesians treat distributions over parameters as epistemic (i.e., about knowledge)
- They do *not* treat them as being doxastic (i.e., about beliefs)
- Priors encode our knowledge before seeing the data
- Posteriors encode our knowledge after seeing the data
- Bayes's rule provides the way to update our knowledge
- People like to pretend models are ontological (i.e., about what exists)

Arbitrariness: Priors vs. Likelihood

- Bayesian analyses often criticized as subjective (arbitrary)
- Choosing priors is no more arbitrary than choosing a likelihood function (or an exchangeability/i.i.d. assumption)
- As George Box famously wrote (1987),

"All models are wrong, but some are useful."

- This does not just apply to Bayesian models!

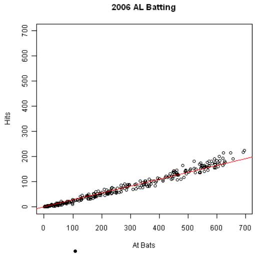
Part IV
Hierarchical Models

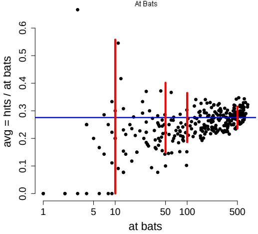
Baseball At-Bats

- For example, consider baseball batting ability.
  - Baseball is sort of like cricket, but with round bats, a one-way field, stationary "bowlers", four bases, short games, and no draws
- Batters have a number of "at-bats" in a season, out of which they get a number of "hits" (hits are a good thing)
- Nobody with higher than 40% success rate since 1950s.
- No player (excluding "bowlers") bats much less than 20%.
- Same approach applies to hospital pediatric surgery complications (a BUGS example), reviews on Yelp, test scores in multiple classrooms, . . .

Baseball Data

- Hits versus at bats for the 2006 American League season
- Not much variation in ability!
- Ignore skill vs. at-bats relation
- Note uncertainty of MLE





Pooling Data

- How do we estimate the ability of a player who we observe getting 6 hits in 10 at-bats? Or 0 hits in 5 at-bats? Estimates of 60% or 0% are absurd!
- Same logic applies to players with 152 hits in 537 at bats.
- *No pooling*: estimate each player separately
- *Complete pooling*: estimate all players together (assume no difference in abilities)
- *Partial pooling*: somewhere in the middle
  - use information about other players (i.e., the population) to estimate a player's ability

Hierarchical Models

- Hierarchical models are principled way of determining how much pooling to apply.
- Pull estimates toward the population mean based on amount of variation in population
  - low variance population: more pooling
  - high variance population: less pooling
- In limit
  - as variance goes to 0, get complete pooling
  - as variance goes to ∞, get no pooling

Hierarchical Batting Ability

- Instead of fixed priors, estimate priors along with other parameters
- Still only uses data once for a single model fit
- Data: y_n, B_n: hits, at-bats for player n
- Parameters: θ_n: ability for player n
- Hyperparameters: α, β: population mean and variance
- Hyperpriors: fixed priors on α and β (hardcoded)

Hierarchical Batting Model (cont.)

$$\begin{array}{rcl} y_n & \sim & \text{Binomial}(B_n, \theta_n) \\ \theta_n & \sim & \text{Beta}(\alpha, \beta) \\ \dfrac{\alpha}{\alpha+\beta} & \sim & \text{Uniform}(0, 1) \\ (\alpha+\beta) & \sim & \text{Pareto}(1.5) \end{array}$$

- Sampling notation syntactic sugar for:

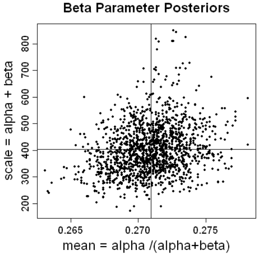
  $p(y, \theta, \alpha, \beta) = \text{Pareto}(\alpha+\beta|1.5) \prod_{n=1}^{N} \left(\text{Binomial}(y_n|B_n, \theta_n)\ \text{Beta}(\theta_n|\alpha, \beta)\right)$

- Pareto provides power law: $\text{Pareto}(u|\alpha) \propto \frac{\alpha}{u^{\alpha+1}}$

- Should use more informative hyperpriors!

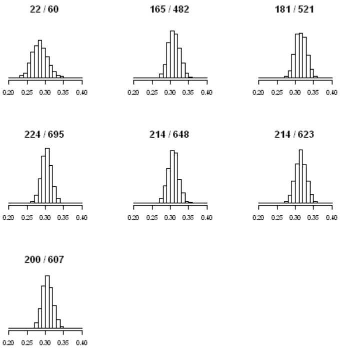
Hierarchical Prior Posterior

- Draws from posterior (crosshairs at posterior mean)
- Prior population mean: 0.271
- Prior population scale: 400
- Together yield prior std dev of 0.022
- Mean is better estimated than scale (typical)



Posterior Ability (High Avg Players)

- Histogram of posterior draws for high-average players
- Note uncertainty grows with lower at-bats



Multiple Comparisons

- Who has the highest ability (based on this data)?
- Probabilty player n is best is

Average	*At-Bats*	Pr[best]
.347	521	0.12
.343	623	0.11
.342	482	0.08
.330	648	0.04
.330	607	0.04
.367	60	0.02
.322	695	0.02

- No clear winner—sample size matters.
- In last game (of 162), Mauer (Minnesota) edged out Jeter (NY)

End (Section 1)